Efficient Mechanical System Optimization Using Two-Point Diagonal Quadratic Approximation in the Nonlinear Intervening Variable Space

Min-Soo Kim
Research Professor, Center of Innovative Design Optimization Technology, Hanyang University, Seoul 133-791, Korea
Jong-Rip Kim, Jae-Young Jeon
Graduate Research Assistant, Center of Innovative Design Optimization Technology, Hanyang University, Seoul 133-791, Korea
Dong-Hoon Choi*
Director, Center of Innovative Design Optimization Technology, Hanyang University, Seoul 133-791, Korea

For efficient mechanical system optimization, a new two-point approximation method is presented. Unlike the conventional two-point approximation methods such as TPEA, TANA, TANA-1, TANA-2 and TANA-3, this introduces the shifting level into each exponential intervening variable to avoid the lack of definition of the conventional exponential intervening variables due to zero- or negative-valued design variables. Then a new quadratic approximation whose Hessian matrix has only diagonal elements of different values is proposed in terms of these shifted exponential intervening variables. These diagonal elements are determined in a closed form that corrects the typical error in the approximate gradient of the TANA series due to the lack of definition of exponential type intervening variables and their incomplete second-order terms. Also, a correction coefficient is multiplied to the pre-determined quadratic term to match the value of approximate function with that of the previous point. Finally, in order to show the numerical performance of the proposed method, a sequential approximate optimizer is developed and applied to solve six typical design problems. These optimization results are compared with those of TANA-3. These comparisons show that the proposed method gives more efficient and reliable results than TANA-3.

Key Words: Two-Point Approximation, Sequential Approximate Optimization

1. Introduction

In the 1970's, Schmit and his coworkers introduced suitable approximation concepts (Schmit and Farshi, 1974; Schmit and Miura, 1976; Schmit and Fleury, 1980). They combined the now familiar techniques of intervening variable definition, explicit approximation, reduced basis and design variable linking as well as constraint deletion and regionalization. In the 1980's, most of approximations were based on function and gradient information at a single point and constructed by the first-order Taylor series expansion at this point, which are the linear, reciprocal and conservative approximations (Schmit and Fleury, 1980). This is very popular because the function and its derivative values are

* Corresponding Author,
E-mail: dhchoi@email.hyu.ac.kr
TEL: +82-2-2290-0478; **FAX**: +82-2-2291-4070
Center of Innovative Design Optimization Technology, Hanyang University, Haengdang-Dong, Sungdong-Ku, Seoul 133-791, Korea. (Manuscript **Received** February 1, 2001; **Revised** July 5, 2001)

always required in the most of optimization algorithms, so no additional computation is involved in constructing the approximate functions. Although these approximation works effectively for stress and displacement functions, the truncated error of them might be large.

In the 1990's, in order to make full use of the known information to construct approximate functions, many multi-point approximations have been developed (Wang and Grahdhi, 1995, 1996a, 1996b; Fadel et al., 1990; Xu and Grandhi, 1998). Among them, two-point approximation methods, first introduced by Fadel et. al. (1990), are widely used for their simplicity. They considered intervening variables in terms of exponentials, which were computed by matching the gradient of approximate function with the previous design point's exact value. Based on these exponential intervening variables, Wang and Grandhi (1995) developed an improved two-point approximation using both function and gradient information of two data points, which were called TPEA-change, TANA, TANA-1 and TANA-2. Recently, Xu and Grandhi (1998) developed TANA-3 having diagonal and changeable Hessian matrix in order to avoid the computational burden of solving $n+1$ nonlinear equations for each function in TANA-2. Owing to its changeable quadratic terms, however, TANA-3 may make a point of inflection between two points used for approximation although the original function is convex between them. This false approximation can retard the convergence of sequential approximation optimization (SAO). Also, TANA-3 may falsely give non-zero derivative values with respect to some design variables on which the original function is not dependant.

This paper presents a new two-point approximation. Unlike other two-point approximations such as TPEA (Fadel et al., 1990), TANA, TANA-1, TANA-2 (Wang and Grandhi, 1995) and TANA-3 (Xu and Grandhi, 1998), this introduces the shifting level into each exponential intervening variable to avoid the lack of definition of the conventional exponential intervening variables due to zero- or negative-valued design variables. Then a new quadratic approximation whose Hessian matrix has only diagonal elements of different values is proposed in terms of these shifted exponential intervening variables. These diagonal elements are determined in a closed form that corrects the typical error in the approximate gradient of the TANA series due to the lack of definition of exponential type intervening variables and their incomplete second-order terms. Also, a correction coefficient is multiplied to the pre-determined quadratic term to match the value of approximate function with that of the previous point.

Section 2 reviews the typical two-point approximations such as TPEA, TANA, TANA-1, TANA-2 and TANA-3. Section 3 fully describes the proposed two-point diagonal quadratic approximation (TDQA). Section 4 describes the computational procedure of SAO combined with the TDQA. Section 5 shows the numerical performance of the SAO combined with TDQA. Finally, the concluding remarks are presented in Sec. 6.

2. Review of the Two-Point Approximations

In this section, we describe the mathematical details of the previous two-point approximations such as TPEA, TANA, TANA-1, TANA-2 and TANA-3 in order to better explain the proposed method. The known design points are denoted as $\boldsymbol{x}_1(x_{1,1}, x_{2,1}, \cdots, x_{n,1})$ and $\boldsymbol{x}_2(x_{1,2}, x_2, \cdots, x_{n,2})$ where the function and gradient information are available. Here n is the number of design variables. The function $\tilde{g}(\boldsymbol{x})$ denotes the approximate function based on two-point approximation, which is expanded at the current design point $\boldsymbol{x}_2$ and uses the values of function and/or derivatives of two design points.

2.1 Two-point exponential approximation (TPEA)

Fadel et al. (1990) first developed a two-point exponential approximation. It is a linear Taylor approximation in terms of the intervening variables

$$y_i = x_i^{p_i},\ i=1,2,\cdots,n \tag{1}$$

where the exponent p_i for each design variable is evaluated by matching the derivatives of the approximate function with those of the exact function at the previous design point. p_i is obtained in a closed form solution, that is

$$p_i = 1 + \ln\left[\frac{\partial g(\boldsymbol{x}_1)}{\partial x_i}\bigg/\frac{\partial g(\boldsymbol{x}_2)}{\partial x_i}\right]\bigg/\ln(x_{i,1}/x_{i,2}) \tag{2}$$

The approximate function is given in terms of the original variables x_i as

$$\tilde{g}(\boldsymbol{x}) = g(\boldsymbol{x}_2) + \sum_{i=1}^{n}\frac{\partial g(\boldsymbol{x}_2)}{\partial x_i}\frac{x_{i,2}^{1-p_i}}{P_i}(x_i^{p_i} - x_{i,2}^{p_i}) \tag{3}$$

In this approximation, the value of p_i is limited from -1 to +1. However, Wang and Grandhi (1995) removed the limitation of p_i for better adaptability for different structural problems, which was called as TPEA-change method.

2.2 Two-point adaptive nonlinear approximations (TANA)

Wang and Grandhi (1995) proposed TANA method using adaptive intervening variables as $y_i = x_i^r$, $i=1, 2, \cdots, n$, where r represents the nonlinearity index, which is different at each iteration but is the same for all variables. The nonlinearity index was determined by matching the function value of the previous design point. Also, In order to utilize more information in constructing better approximation, Wang and Grandhi (1995) proposed the following two approximation methods (TANA-1 and TANA-2) to combine TPEA-change and TANA methods.

In TANA-1 approach, the approximation is expanded at the previous design point $\boldsymbol{x}_1$ instead of the current point $\boldsymbol{x}_2$ to reproduce the most recent information exactly, that is

$$\tilde{g}(\boldsymbol{x}) = g(\boldsymbol{x}_1) + \sum_{i=1}^{n}\frac{\partial g(\boldsymbol{x}_1)}{\partial x_i}\frac{x_{i,1}^{1-p_i}}{p_i}(x_i^{p_i} - x_{i,1}^{p_i}) + \varepsilon_1 \tag{4}$$

where ε_1 is a constant, representing the residue of the first-order Taylor approximation in terms of the intervening variables y_i. To evaluate p_i and ε_1, the approximate function value and its derivatives are matched with those of exact function at the current point $\boldsymbol{x}_2$. But TANA-1 is the same as TPEA in the result.

In the TANA-2 approach, the approximation is written by expanding the function at $\boldsymbol{x}_2$ and includes the second-order Taylor series effects, in which the Hessian matrix has only diagonal elements of the same value ε_2.

$$\tilde{g}(\boldsymbol{x}) = g(\boldsymbol{x}_2) + \sum_{i=1}^{n}\frac{\partial g(\boldsymbol{x}_2)}{\partial x_i}\frac{x_{i,2}^{1-p_i}}{p_i}(x_i^{p_i} - x_{i,2}^{p_i}) + \frac{1}{2}\varepsilon_2\sum_{i=1}^{n}(x_i^{p_i} - x_{i,2}^{p_i})^2 \tag{5}$$

In order to get $n+1$ unknown constants (p_i and ε_2), $n+1$ equations are required. The n equations are obtained by matching $\partial\tilde{g}(\boldsymbol{x}_1)/\partial x_i = \partial g(\boldsymbol{x}_1)/\partial x_i$, $i=1, 2, \cdots, n$. The $(n+1)^{th}$ equation is obtained by matching $\tilde{g}(\boldsymbol{x}_1) = g(\boldsymbol{x}_1)$. Then, $n+1$ unknown constants can be obtained by solving these $n+1$ coupled nonlinear equations.

TANA-1 and TANA-2 had either incomplete matching at two design points or the additional solving of equations that was needed to get some parameters. Recently Xu and Grandhi (1998) developed TANA-3, which was the incomplete second order Taylor series expansion in terms of the intervening variables, in which Hessian matrix was diagonal and changeable. This approximation method used the intervening variables given in Eq. (1). The approximation was represented by expanding the function at $\boldsymbol{x}_2$.

$$\tilde{g}(\boldsymbol{x}) = g(\boldsymbol{x}_2) + \sum_{i=1}^{n}\frac{\partial g(\boldsymbol{x}_2)}{\partial x_i}\frac{x_{i,2}^{1-p_i}}{p_i}(x_i^{p_i} - x_{i,2}^{p_i}) + \frac{1}{2}\varepsilon_3(\boldsymbol{x})\sum_{i=1}^{n}(x_i^{p_i} - x_{i,2}^{p_i})^2 \tag{6}$$

specifying

$$\varepsilon_3(\boldsymbol{x}) = H\bigg/\left[\sum_{i=1}^{n}(x_i^{p_i} - x_{i,1}^{p_i})^2 + \sum_{i=1}^{n}(x_i^{p_i} - x_{i,2}^{p_i})^2\right] \tag{7}$$

where p_i and H are constants to be obtained in a closed form solution to match $\tilde{g}(\boldsymbol{x}_1) = g(\boldsymbol{x}_1)$ and . $\nabla\tilde{g}(\boldsymbol{x}_1) = \nabla g(\boldsymbol{x}_1)$. The value of p_i is equal to Eq. (2) and H is

$$H = 2\cdot\left[g(\boldsymbol{x}_1) - g(\boldsymbol{x}_2) - \sum_{i=1}^{n}\frac{\partial g(\boldsymbol{x}_2)}{\partial x_i}\frac{x_{i,2}^{1-p_i}}{p_i}(x_{i,1}^{p_i} - x_{i,2}^{p_i})\right] \tag{8}$$

Special provisions needed to be made when the ratios in the numerator or denominator in Eq. (2) are negative or the denominator is close to 1. In the first case Xu and Grandhi (1998) assigned a specialized value (1 or -1) to p_i. While in the second case they considered the optimization iterations near the convergence domain and the design variables being hardly changed. Thus, they assigned a specialized value (1 or -1) to p_i, too. On the other side, the magnitude of p_i may be large and deteriorate the approximation. Thus, they put a bound value on p_i when the magnitude of p_i is greater than bound value. It is rounded down to the bound value. They recommended the bound value as $\mathrm{sign}(p_i)\cdot 5$.

Although TANA-3 can overcome the computational burden of TANA-2 that needs additional solving of $n+1$ equations for each function, it has the following three problems due to its changeable quadratic terms.

- TANA-3 may make a point of inflection between two points used for approximation although the original function is convex between them.
- The approximation accuracy is not guaranteed when the ratios in the numerator or denominator in Eq. (2) are negative, even though Xu and Grandhi provide the special provisions for these cases.
- TANA-3 may falsely give non-zero derivative values with respect to some design variables on which the original function is not dependent. In other words, although $g(\boldsymbol{x})$ do not depend on x_i, the following approximate derivative may not be zero-value because $\varepsilon_3(\boldsymbol{x})\neq 0$ or $\partial\varepsilon_3(\boldsymbol{x})/\partial x_i\neq 0$.

$$\frac{\partial\tilde{g}(\boldsymbol{x})}{\partial x_i}=\frac{1}{2}\frac{\partial\varepsilon_3(\boldsymbol{x})}{\partial x_i}\sum_{i=1}^{n}(x_i^{p_i}-x_{i,2}^{p_i})^2+\varepsilon_3(\boldsymbol{x})(x_i^{p_i}-x_{i,2}^{p_i})\cdot p_i x_i^{p_i-1} \tag{9}$$

These defects can be similarly occurred in TANA-2 because its correction coefficient ε_2 can be considered as a simplified form of $\varepsilon_3(\boldsymbol{x})$. And TANA-2 cannot represent curvatures of different signs along the intervening variable coordinates due to its constant diagonal Hessian term ε_2.

3. Two-Point Diagonal Quadratic Approximation (TDQA)

3.1 Basic concept

This study presents a new two-point approximation, which is called as a Two-point Diagonal Quadratic Approximation (TDQA). This introduces the shifting level into the exponential intervening variables as

$$y_i=(x_i+c_i)^{p_i},\ i=1,2,\cdots,n \tag{10}$$

where c_i is the shifting level for the i^{th} design variable. The unknown exponents p_i are determined in the same way of TANA-3.

$$p_i=1+\ln\left[\frac{\partial g(\boldsymbol{x}_1)}{\partial x_i}\Big/\frac{\partial g(\boldsymbol{x}_2)}{\partial x_i}\right]\Big/\ln\left[(x_{i,1}+c_i)/(x_{i,2}+c_i)\right] \tag{11}$$

As we described in Sec. 2.2, the value of p_i can be inappropriately determined when the ratios in the numerator or denominator in Eq. (11) are negative- or zero-values. For these cases, the detailed special provisions are discussed in Sec. 3.2.

In this section, we propose a new quadratic model in terms of the shifted intervening variables as

$$\tilde{g}(\boldsymbol{x})=g(\boldsymbol{x}_2)+\sum_{i=1}^{n}\frac{\partial g(\boldsymbol{x}_2)}{\partial y_i}(y_i-y_{i,2})+\sum_{i=1}^{n}G_i(y_i-y_{i,2})^2 \tag{12}$$

The proposed approximation expands the function at $\boldsymbol{x}_2$. The diagonal component of the Hessian is defined as

$$G_i=\frac{1}{2(y_{i,1}-y_{i,2})}\left(\frac{\partial g(\boldsymbol{y}_1)}{\partial y_i}-\frac{\partial g(\boldsymbol{y}_2)}{\partial y_i}\right) \tag{13}$$

in order to correct the error of the approximate gradient at the previous design point. As the exponent p_i is determined to match $\partial\tilde{g}(\boldsymbol{y}_1)/\partial y_i=\partial g(\boldsymbol{y}_1)/\partial y_i$, the appropriate p_i makes $G_i=0$. Finally, in order to match $\tilde{g}(\boldsymbol{x}_1)=g(\boldsymbol{x}_1)$, the correction coefficient η is determined as

$$\eta=\left[g(\boldsymbol{x}_1)-g(\boldsymbol{x}_2)-\sum_{i=1}^{n}\frac{\partial g(\boldsymbol{x}_2)}{\partial y_i}(y_{i,1}-y_{i,2})\right]\Big/\sum_{i=1}^{n}G_i(y_{i,1}-y_{i,2})^2 \tag{14}$$

Consequently, the final form of the proposed approximation can be represented as

$$\tilde{g}(\boldsymbol{x})=g(\boldsymbol{x}_2)+\sum_{i=1}^{n}\frac{\partial g(\boldsymbol{x}_2)}{\partial y_i}(y_i-y_{i,2})+\eta\sum_{i=1}^{n}G_i(y_i-y_{i,2})^2 \tag{15}$$

Now we examine the problems of TANA-3 mentioned at the end of Sec. 2.

- In comparison with Eq. (9), the approximate derivatives of TDQA with respect to some design variables, of which a function is independent, are

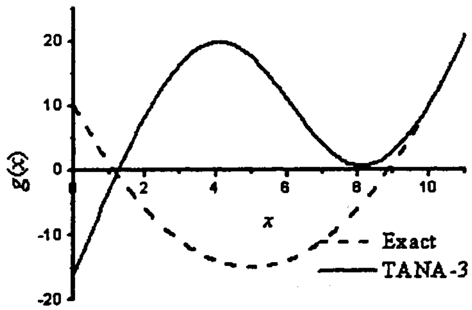

Fig. 1 TANA-3 approximation to a function of x

always zero because $\partial g(y_1)/\partial y_i=0$ and $\partial g(y_2)/\partial y_i=0$, therefore $G_i=0$ from Eq. (13).

• Next, in order to better understand the difference of the quadratic correction terms between TANA-3 and TDQA, consider the following function

$g(x)=(x-5)^2-15$ with $x_1=1.2$ and $x_2=11.0$

which is convex between two points, is approximated. The function may be so simple but one can gain some insight into the difference between TANA-3 and TDQA clearly. The function values and the derivatives are $g(x_1)=-0.56$, $g(x_2)=21.0$, $dg/dx\,|_{x_1}=-7.6$ and $dg/dx\,|_{x_2}=12.0$ respectively. As$(dg/dx)_{x_1}/(dg/dx)_{x_2}<0$, p is forced to be 1. Also, we get $H=192.08$ in TANA-3 and $G=\eta=1$ in TDQA. Thus the approximate function in TDQA is the identical one,

$$\tilde{g}_{TDQA}=21+12(x-11)+(x-11)^2=(x-5)^2-15$$

But that in TANA-3 becomes

$$\tilde{g}_{TANA\text{-}3}=21+12(x-11)+96.04(x-11)^2/[(x-1.2)^2+(x-11)^2]$$

This approximate function $\tilde{g}_{TANA-3}$ is plotted on Fig. 1, together with the original function. Figure 1 shows that the approximation of TANA-3 is very poor. This is principally due to the quadratic term $\varepsilon_3(x)$ in TANA-3.

3.2 Numerical considerations for constructing TDQA

In this section we describe some guidelines to determine the four parameters of c_i, p_i, G_i and η in the TDQA.

3.2.1 Determination of the shifting level c_i

If the current design variable x_i is less than a small positive real value ζ, then $c_i=|x_i^L|+1$ is used, where x_i^L denotes the lower bound for the i^{th} design variable. Otherwise $c_i=0$. This shifting level can avoid the singularity of the approximate derivatives in the neighborhood of $x_i=0$ and the fundamental difficulties of other two-point approximations occurred for $x_i<0$. The value of ζ is recommended as 1×10^{-3}.

3.2.2 Provisions for the exponent p_i

Special provisions need to be made when the ratios in the numerator or denominator in Eq. (11) are negative or zero. When the numerator is less than or equal to zero, we assign $p_i=1$. This represents that a quadratic approximation is taken in terms of x_i because of the definition of G_i. Also, when the denominator goes to 1 such as $|(x_{i,1}+c_i)/(x_{i,2}+c_i)-1|\le\varepsilon$, $p_i^k=p_i^{k-1}$ is assigned with $p_i^0=1$. The superscript κ is the number of iterations in SAO. The value of ε is recommended as $\varepsilon=1\times10^{-2}$.

The magnitude of p_i may be large and deteriorate the approximation. Thus, we put a bound value p_{max} on p_i when the magnitude of p_i is greater than p_{max}. It is rounded up and down to $(-p_{max}, p_{max})$. We assign $p_{max}=5$.

3.2.3 Provision for the diagonal term G_i in the Hessian

When the denominator goes to zero in Eq. (13), the value of G_i becomes infinite. We believe that this deteriorates the approximation of a function. Thus, $G_i=0$ is assigned when the denominator $|y_{i,1}-y_{i,2}|$ is less than or equal to $\varepsilon|y_{i,2}|$. The value of ε is recommended as $\varepsilon=1\times10^{-2}$.

3.2.4 Provision for the correction coefficient η

The correction coefficient η can be a large value when the denominator of Eq. (14) becomes a small value. However, the larger η deteriorates the approximate gradient $\nabla\tilde{g}(\boldsymbol{x})$, even though it ameliorates the approximate function $\tilde{g}(\boldsymbol{x})$. Thus, we check the following condition, Eq. (16), before determining the correction coefficient η.

$$\left|\sum_{i=1}^{n} G_i (y_{i,1}-y_{i,2})^2\right| > \varepsilon \cdot \left| g(x_1)-g(x_2)-\sum_{i=1}^{n}\frac{\partial g(x_2)}{\partial y_i}(y_{i,1}-y_{i,2})\right| \quad (16)$$

where the value of ε is recommended as $\varepsilon = 1\times 10^{-2}$. If this condition is satisfied, then the correction coefficient η is used. Otherwise, $\eta=1$ is used. In other words, the exactly estimated correction coefficient from Eq. (14) is used only if the predetermined quadratic term is greater than 1 % of the linear term in the approximate function $\tilde{g}(\boldsymbol{x})$. Otherwise, we neglect the function value matching at the previous design point because the error is less than 1 %.

4. Computational Procedure of Sequential Approximate Optimization with TDQA

In order to use the TDQA in the sequential approximate optimization (SAO), the computational procedure is described as:

Step 0. Evaluate function and gradient values of objective $f(\boldsymbol{x})$ and constraint functions $g_j(\boldsymbol{x})$, $j=1, \cdots, m$, for the initial design $\boldsymbol{x}_0$. Set $\kappa=0$.

Step 1. If $\kappa=0$, construct the function approximations using conservative method and go to Step 2. Otherwise, construct them using TDQA and go to Step 3.

Step 2. Solve the following approximate optimization problems with 40 percent move limit: minimize $\tilde{f}(\boldsymbol{x})$ subject to $\tilde{g}_j(\boldsymbol{x})\leq 0$, $j=1, \cdots, m$ and $x_i^L \leq x_i \leq x_i^U$ for $i=1, \cdots, n$. Let $\tilde{\boldsymbol{x}}_k^*$ be the approximate optimum. Go to Step 4.

Step 3. Solve the approximate optimization problems, with the initial design $\tilde{\boldsymbol{x}}_{k-1}^*$, without any move limit: minimize $\tilde{f}(\boldsymbol{x})$ subject to $\tilde{g}_j(\boldsymbol{x})\leq 0$, $j=1,\cdots, m$ and $x_i^L \leq x_i \leq x_i^U$ for $i=1,\cdots, n$. Let $\tilde{\boldsymbol{x}}_k^*$ be the approximate optimum. Go to Step 4.

Step 4. Evaluate the exact function values at the approximate optimum $\tilde{\boldsymbol{x}}_k^*$. If the convergence criteria of $|f(\tilde{\boldsymbol{x}}_k^*)-f(\boldsymbol{x}_k)| \leq \tau_1 |f(\boldsymbol{x}_k)|$ and $g_j(\tilde{\boldsymbol{x}}_k^*)\leq \tau_2$ for $j=1, \cdots, m$ are satisfied, then the optimization is terminated. Otherwise, go to Step 5.

Step 5. Evaluate the gradient values of objective and constraints at $\tilde{\boldsymbol{x}}_k^*$ and update the design variable $\boldsymbol{x}_{k+1}=\tilde{\boldsymbol{x}}_k^*$. Return to Step 1 with $\kappa=\kappa+1$.

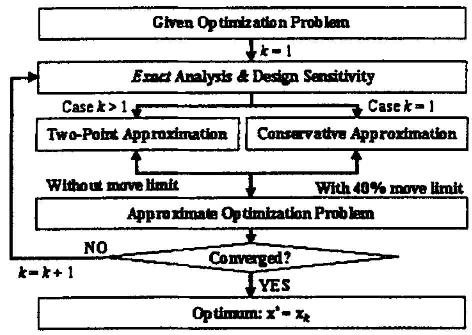

Fig. 2 Flow chart of SAO process

In Steps 2 and 3, the approximate optimization problem can be solved using any constrained optimizers. This study uses the sequential quadratic programming (Vanderplaats, 1984). A flow chart of this process is shown in Fig. 2.

5. Numerical Examples

In order to examine the numerical performance of the TDQA, a sequential approximate optimizer having option of two approximation methods such as TDQA and TANA-3, is developed based on the computational procedures described in Sec. 4. In these comparisons, TANA-2 is not included, because it requires additional solving of $n+1$ equations for each function.

The six test problems considered include two mechanical system designs, four design cases of plane ten-bar truss (Haug and Arora, 1979). Two mechanical system design problems are the welded beam design (Reklatis et al., 1983) and the coil spring design (Haug and Arora, 1979). In the coil spring design problem, the error of both approximation methods, described in Sec. 2 and 3, is clearly elucidated. In the four design cases of ten-bar truss, the effect of using the shifted intervening variable is numerically shown. The same convergence tolerances are taken as $\tau_1=1\times 10^{-3}$ and $\tau_2=1\times 10^{-3}$ for all test problems.

5.1 Welded beam design problem

This design problem has been widely used in the Reklaitis et al. (1983). The design objective of this welded beam design (Fig. 3) is to minimize

Table 1 Comparison of optimization results for the welded beam design

	Initial design	TDQA	TANA-3
x_1	1.0	0.2409	0.2444
x_2	7.0	6.3239	6.2317
x_3	4.0	8.3285	8.3010
x_4	2.0	0.2443	0.2444
f	15.8138	2.3946	2.3860
g_{max}	-0.9048	-0.0027	0.0000
Iterations	-	10	11

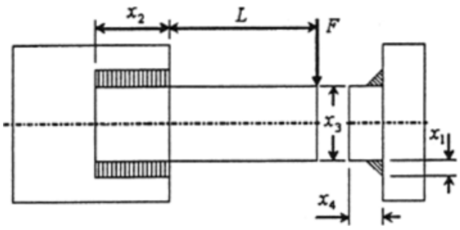

Fig. 3 Welded beam

Fig. 4 Coil spring

the overall welding cost while satisfying constraints on maximum shear stress in weld (g_1), maximum normal stress in beam (g_2), bar buckling load (g_3), minimum deflection of bar end (g_4) and geometric restriction between weld thickness and bar thickness (g_5). The design variables are the weld thickness x_1, the weld length x_2, the bar width x_3 and the bar thickness x_4. The initial design is taken as $\boldsymbol{x}_0=(1, 7, 4, 2)^T$. The constraints 1, 2, 3 and 5 are active at the optimum. The optimum is known as $f(\boldsymbol{x}^*)=2.3811$ and $\boldsymbol{x}^*=(0.2444, 6.2187, 8.2915, 0.2444)^T$.

The optimization results are listed in Table 1, which shows that both methods such as TDQA and TANA-3 can successfully converge to the similar optimum, even though no artificial move limit strategy is employed.

Table 2 Comparison of optimization results for the coil spring design

	Initial design	TDQA	TANA-3
x_1	1.0	0.0529	0.0584
x_2	2.0	0.3863	0.5417
x_3	3.0	9.7938	5.2745
f	10.0	0.0127	0.0134*
g_{max}	1.0	0.0000	0.0006
Iterations	-	11	7

* Prematurely converged.

5.2 Tension/compression spring design problem

This problem is to minimize the weight of a tension/compression spring (shown in Fig. 4) while satisfying constraints on minimum deflection, shear stress, surge frequency, limit on outside diameter and on design variables. The design variables for this problem are the wire diameter x_1, mean coil diameter x_2 and number of active coils x_3. The mathematical formulation is represented as:

minimize $f(\boldsymbol{x})=(x_3+2)x_2x_1^2$

subject to $g_1(\boldsymbol{x})=1-\dfrac{x_2^3x_3}{71875x_1^4}\leq 0$

$$g_2(\boldsymbol{x})=\frac{4x_2^2-x_1x_2}{12566(x_2x_1^3-x_1^4)}+\frac{1}{5108x_1^2}-1\leq 0$$

$$g_3(\boldsymbol{x})=1-\frac{140.45x_1}{x_2^2x_3}\leq 0$$

$$g_4(\boldsymbol{x})=\frac{x_2+x_1}{1.5}-1\leq 0$$

One may refer to (Arora, 1989: pp. 451-453) for detailed formulation on this example. In this numerical test, the initial design and the lower and upper limits on design variables are taken as $\boldsymbol{x}_0=(1, 2, 3)^T$, $\boldsymbol{x}^L=(0.05, 0.05, 1)^T$ and $\boldsymbol{x}^U=(5, 5, 15)^T$. Constraints 1 and 2 are active at the optimum. The optimum is known as $f(\boldsymbol{x}^*)=0.01268$.

The optimization results are listed in Table 2, which show that TDQA gives better result than TANA-3. Then, we trace the convergence path of TANA-3. At the 7^{th} iteration of TANA-3, some interesting results are observed. We believe that these results enable one to clearly understand the difference of both approximation methods'

Table 3 Comparison of approximate derivatives at $\alpha=0.17$

	$\partial g_1(x)/\partial x_1$	$\partial g_1(x)/\partial x_2$	$\partial g_1(x)/\partial x_3$	$\partial g_1(x)/\partial \alpha$
Exact value	23.52	-1.27	-0.35	2.57
TDQA	19.18	-1.53	-0.54	2.42
TANA-3	-1.75	-1.92	-0.65	1.07

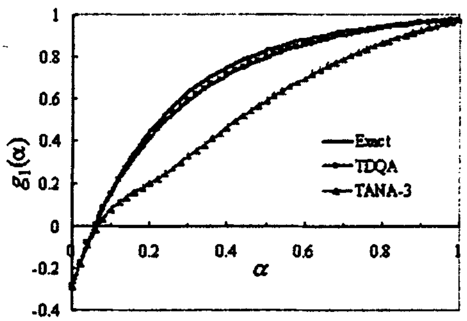

Fig. 5 Comparison of the approximate function values along the vector s

quadratic terms. Now, we examine the accuracies of $\tilde{g}_1(x)$ and $\nabla\tilde{g}_1(x)$ of the 1^{st} inequality constraint approximated from TANA-3 and TDQA, between the 6^{th} and 7^{th} points such as $x_6=(0.1693, 1.0183, 1.5922)^T$ and $x_7=(0.0931, 1.5673, 1.8075)^T$ because this constraint is only active at the 7^{th} iteration of TANA-3. The exponents p_i ($i=1$, 2, 3) of TANA-3 are bounded as (-5, 5, 5) because their values evaluated by Eq. (2) are exceeded them. Also, the value of H is evaluated as -1.04726 using Eq. (8). Then, TDQA can obtain the same exponents p_i because of $c_i=0$. In the quadratic terms of TDQA, G_i ($i=1, 2, 3$) and η are obtained as $(0, -0.00396, -0.0071)^T$ and 0.85968 using Eqs. (13)~(14).

Let the direction vector to be $s=x_6-x_7$. Then the exact and approximate function values are evaluated at $x=x_7+\alpha\cdot s$ in the range $\alpha=(0, 1)$. Figure 5 shows the function values. Although both the approximate function values are quite well matched to the exact function value at two end points ($\alpha=0$ and $\alpha=1$), it is noted that TDQA gives nearly exact values through the interval (0, 1) but TANA-3 has several points of inflection in the range [0, 1]. As you can see, both methods use the same exponents. Thus this difference is caused by only their quadratic terms.

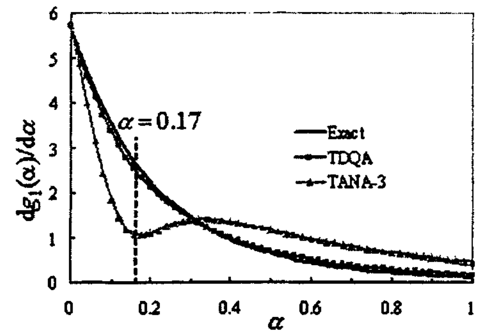

Fig. 6 Comparison of the approximate gradient values along the vector s

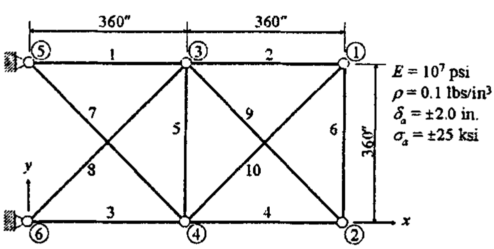

Fig. 7 Ten-bar truss

For more detailed comparisons, the directional derivatives are examined. Figure 6 compares the directional derivative of the two approximate functions, which are defined as $dg_1(x)/d\alpha=\nabla g_1(x)\cdot s$. Figure 6 shows serious approximation error in TANA-3, even though it gives the exact value at the current point ($\alpha=0$). Now, we compare the approximate derivatives with the exact values at $\alpha=0.17$, which lists in Table 3. This comparison shows that the serious error of TANA-3 is caused by the value of $\partial\tilde{g}_1(x)/\partial x_1$ whose sign is not matched with that of exact value. The mathematical reason for this phenomenon is clearly described in the end of Sec. 2.2. We believe that this difference of both approximation methods is caused by the value of H and G_i ($i=1$, 2, 3). Consequently, TANA-3 gives bad search directions during the numerical optimization process.

5.3 Ten-bar truss design problem: case-1

This problem (Fig. 7) has been used extensively in the literature (Haug and Arora, 1979). The

Table 4 Comparison of optimization results for the ten-bar truss design: case-1

	Initial design	TDQA	TANA-3
x_1	1.0	7.9998	7.9545
x_2	1.0	0.0004	0.0624
x_3	1.0	8.0001	8.0427
x_4	1.0	3.9998	3.9523
x_5	1.0	0.0001	0.0001
x_6	1.0	0.0004	0.0624
x_7	1.0	5.6564	5.7204
x_8	1.0	5.6565	5.5934
x_9	1.0	5.6565	5.5933
x_{10}	1.0	0.0003	0.0925
f	419.64	1583.97	1588.15
g_{max}	7.19	0.0001	0.0007
Iterations	-	10	14

Table 5 Comparison of optimization results for the ten-bar truss design: case-2

	Initial design	TDQA	TANA-3
x_1	1.0	29.5333	29.8105
x_2	1.0	0.0001	0.0001
x_3	1.0	22.9959	23.1649
x_4	1.0	15.3412	15.4185
x_5	1.0	0.0001	0.0001
x_6	1.0	0.2392	0.6497
x_7	1.0	7.6663	7.6661
x_8	1.0	20.4640	20.4560
x_9	1.0	21.7987	21.3552
x_{10}	1.0	0.0001	0.0001
f	419.64	4993.92	5004.55
g_{max}	18.70	0.0000	0.0010
Iterations	-	12	12

loading consists of 100 kips applied in the negative ydirection at nodes 2 and 4. The allowable stress for each element is $\sigma_a=25$ ksi in tension or compression, the lower and upper limits for each element are 0.0001 in^2 and 50 in^2. The mass density is $\rho=0.1$ lb/in^3, Young's modulus is $E=10^7$ psi. The initial design is taken as 1.0 in^2 for each element.

The optimization results are listed in Table 4. In this problem, stress constraints for elements 1, 3, 4 and 7-9 and the lower limits on elements 2, 5, 6 and 10 are active at the optimum design. Table 4 shows that TANA-3 is more sensitive to the lower limits on elements. However, it is noted

Table 6 Comparison of optimization results for the ten-bar truss design: case-3

	Initial design	TDQA	TANA-3
x_1	1.0	5.9871	5.9874
x_2	1.0	0.0208	0.0290
x_3	1.0	10.0128	10.0130
x_4	1.0	3.9867	3.9862
x_5	1.0	0.0001	0.0001
x_6	1.0	2.0133	2.0144
x_7	1.0	8.5034	8.5037
x_8	1.0	2.8104	2.8113
x_9	1.0	5.6381	5.6371
x_{10}	1.0	0.0286	0.0307
f	419.64	1657.26	1657.70
g_{max}	7.37	0.0002	0.0007
Iterations	-	12	14

that TDQA can converge to the lower limits on elements. This shows the effectiveness of the shifted intervening variables in TDQA. When we traced the optimization history, the premature convergence was caused by the same phenomenon shown in Fig. 6.

5.4 Ten-bar truss design problem: case-2

The description of this problem is the same as for the ten-bar truss design problem case-1, except that the displacement for each node is constrained in $\delta_a=\pm 2.0$ in.

The optimization results are listed in Table 5. The downward displacement constraint at node 2 and the minimum size constraints for elements 2, 5 and 10 are active at the optimum. Both approximation methods are successfully converged to nearly the same optimum, although TDQA gives better results than TANA-3.

5.5 Ten-bar truss design problem: case-3

The description of this problem is the same as for ten-bar truss design problem case- 1, except the loading condition. In this problem, the loading consists of 150 kips applied in the negative y-direction at nodes 2 and 4, and 50 kips applied in the positive y-direction at nodes 1 and 3.

The optimization results are listed in Table 6. The stress constraints on elements 2, 5, and 10 are active at the optimum. Both approximation

Table 7 Comparison of optimization results for the ten-bar truss design: case-4

	Initial design	TDQA	TANA-3
x_1	1.0	22.9287	23.0235
x_2	1.0	0.0001	0.9974
x_3	1.0	25.3236	25.2012
x_4	1.0	14.2363	14.2246
x_5	1.0	0.0001	0.0001
x_6	1.0	2.0003	2.0002
x_7	1.0	12.7557	12.7519
x_8	1.0	12.1823	12.1201
x_9	1.0	20.1744	20.3548
x_{10}	1.0	0.0001	0.0001
f	419.64	4619.15	4658.67
g_{max}	19.06	0.0010	0.0002
Iterations	-	8	8

methods are successfully converged to nearly the same optimum, while TDQA saves two analyses than TANA-3.

5.6 Ten-bar truss design problem: case-4

The description of this problem is the same as for the ten-bar truss design problem case-3, except that the displacement for each node is constrained in $\delta_a = \pm 2.0$ in.

The optimization results are listed in Table 7. The downward displacement constraint at node 2, stress in element 5, and the lower limit on elements 2 and 10 are active at the optimum. Both approximation methods are successfully converged to nearly the same optimum. However, it is noted that TDQA gives better results than TANA-3 because TDQA gives more active design to the lower limits on elements 2, 5, and 10. This shows the effectiveness of the shifted intervening variables in TDQA.

6. Concluding Remarks

This study presented a new Two-point Diagonal Quadratic Approximation (TDQA) in terms of the exponential intervening variables. This introduced the shifting levels into intervening variables to avoid the numerical difficulties of conventional two-point approximations in the neighborhood of $x_i = 0$ and the critical difficulties of them that could not be used for $x_i < 0$. Also, in this method, a new quadratic form is introduced in the proposed intervening variable space in order to overcome the critical difficulty that other two-point approximation methods did not approximate a convex function due to lack of definition of its intervening variables and did not represent different signed curvatures due to its incomplete quadratic terms.

A sequential approximate optimizer with option of two approximation methods such as TDQA and TANA-3 was developed and applied to two mechanical system designs and four cases of plane ten-bar design problems. In these numerical tests, we numerically show the defect of TANA-3 mathematically shown in Sec. 2 and the role of shifted intervening variables. Also, we compared the performance of the proposed TDQA with those of TANA-3. These comparisons clearly show the superiority of TDQA over TANA-3, which verifies that the TDQA is an effective and efficient two-point approximation.

Acknowledgement

This work was supported by center of Innovative Design Optimization Technology (iDOT), Korea Science and Engineering Foundation and the Agency for Defense and Development Grant No. ADD-00-05-08.

References

Arora, J. S., 1989, *Introduction to Optimum Design*, McGraw-Hill, New York, pp. 489~493.

Fadel, G. M., Riley, M. F. and Barthelemy, J. F. M., 1990, "Two Point Exponential Approximation Method for Structural optimization," *Structural Optimization*, Vol. 2, pp. 117~124.

Haug, E. J. and Arora, J. S, 1979, *Applied optimal Design Mechanical and Structural Systems*, John Wiley & Sons, New York, pp. 242~245.

Reklaitis, G. V., Ravindran, A. and Ragsdell, K. M., 1983, *Engineering Optimization Methods and Applications*, John Wiley & Sons, New

York, pp. 11~15.

Schmit, L. A. and Farshi, B., 1974, "Some Approximation Concepts for Structural Synthesis," *AIAA Journal*, Vol. 12, pp. 692~699.

Schmit, L. A. and Miura, H., 1976, "A New Structural Analysis/Synthesis Capablity -ACCESS 1," *AIAA Journal*, Vol. 14, pp. 661~671.

Schmit, L. A. and Fleury, C., 1980, "Structural Synthesis by Combining Approximation Concepts and Dual Methods," *AIAA Journal*, Vol, 18, pp. 1252~1260.

Vanderplaats, G. N., 1984, *Numerical Optimization Techniques for Engineering with Applications*, McGraw-Hill, New York, pp. 195~199.

Wang, L. P. and Grandhi, R. V., 1995, "Improved Two-Point Function Approximation for Design Optimization," *AIAA Journal*, Vol. 33, No. 9, pp. 1720~1727.

Wang, L. P. and Grandhi, R. V., 1996a, "Multipoint Approximations: Comparisons Using Structural Size, Configuration and Shape Design," *Structural Optimization*, Vol. 12, pp. 177~185.

Wang, L. P. and Grandhi, R. V., 1996b, "Multivariate Hermite Approximation for Design Optimization," *International Journal for Numerical Methods in Engineering*, Vol. 39, pp. 787~803.

Xu, S. and Grandhi, R. V., 1998, "An Effective Two-Point Function Approximation for Design Optimization, " *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 39th Structural, Structural Dynamics, and Materials Conference, Long beach CA, April 20-23*, pp. 2181~2191.